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1994 J. Phys.: Condens. Matter 6 3771

(http://iopscience.iop.org/0953-8984/6/20/016)

J. Phys.: Condens. Matter 6 (1994) 3771–3780. Printed in the UK

Superconductivity in the two-dimensional t–J model at low electron density

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Received 4 March 1994

Abstract. The two-dimensional t–J model at low electron densities is unstable against various forms of electron pairing at low enough temperatures. At parameter values $J/t \ll 1$, the leading instability, although only at very low temperature, is to p-wave pairing similar to the Hubbard model. At values of $J/t > 1$, an instability against d-wave pairing sets in at a higher temperature as found numerically by Dagotto and co-workers. In addition, at values of $J/t > 2$, which are beyond the threshold for a bound state in the low-electron-density limit, a region of predominantly s-wave pairing is found.

1. Introduction

The t–J model was derived many years ago by Bulaevskii and co-workers [1] to describe the strong-coupling limit of the single-band Hubbard model. The study of this model has become very active in recent years due to Anderson's proposal [2] that it was the appropriate model to describe the doped CuO_2 planes that are the key ingredients of the high-T_c cuprates. Later Zhang and Rice [3] elucidated the relationship of the t–J model to a multiband Hubbard description with Cu $3d_{x^2-y^2}$ and O $2p_\sigma$ orbitals. The careful numerical investigation of Hybertsen and co-workers [4] established the parameter values in the mapping of the multiband Hubbard model for the CuO_2 planes into an one-band t–J model, namely $J \sim 0.3t$. In the single-band Hubbard model the mapping to a t–J model is valid only in the strong-coupling limit which leads to values $J \ll t$. In a more general model other values of J/t can occur. A lot of work has been done to clarify analytically the relationship between the t–J and multiband Hubbards, see e.g. [5] and references therein. In this paper we will treat the ratio J/t simply as a parameter to be varied arbitrarily.

The leading pairing instability of the two-dimensional Hubbard model at strong coupling and low electron density was found by Baranov, Kagan and Chubukov [6, 7] to be to a p-wave triplet pairing. Note this instability arises only when higher-order terms are included in the two-particle $\mathbf{T}$-matrix, and occurs only at very low temperatures. In view of the close relationship between the Hubbard and t–J models we expect a similar instability in the latter model when $J \ll t$. Recently, Dagotto and co-workers [8] found by numerical means pairing instabilities as precursors to phase separation. The onset in the numerical studies was at $J = 2t$ which, as reported by Emery, Kivelson and Lin [9, 10], is the threshold for a singlet bound state of two electrons in an empty lattice. In the low-density region, electron density $n \lesssim 0.25$, Dagotto and co-workers [8] found the leading pairing instability when

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0953-8984/94/203771+10$19.50

$J > 2t$ is to singlet s-wave pairing but at a higher density ($n > 0.25$) there was a crossover to $d_{x^2-y^2}$ pairing as the leading instability.

In this paper we will extend the earlier work of Baranov, Kagan and Chubukov [6, 7] on pairing instabilities in the low-density Hubbard model to the case of the t–J model. We are particularly interested in understanding, from an analytic viewpoint, the factors which determine the competition between these three different pairing symmetries, namely triplet p-wave at $J \ll t$ and singlet s and $d_{x^2-y^2}$ pairings when $J \gtrsim t$. A study of the **T**-matrix in the various symmetry channels shows how the pairing instabilities evolve with changing n and J/t. We find good agreement between our analytic approximations and the numerical calculations of the boundary between s and $d_{x^2-y^2}$ pairing with increasing density n at $J \geqslant 2t$. It is interesting to note that if we arbitrarily extend these low-electron-density calculations to the relevant parameter regime for the cuprates ($J/t \sim \frac{1}{2}$, $n \simeq 0.85$), we find a high-temperature instability to $d_{x^2-y^2}$ pairing.

The outline of this paper is as follows. In the next section we examine the form of the **T**-matrix for particle–particle scattering in the low-density limit and obtain the thresholds for a two-particle bound state in both the s- and d-wave singlet channels. In the third section we use these forms of the **T**-matrix to estimate the mean field transition temperature for pairing instabilities in these channels and also in the p-wave triplet channel. We compare our results with the previous work on the Hubbard model and the numerical calculations for the t–J model. The last section is devoted to concluding remarks.

2. The T-matrix in the particle–particle channel at low electron density

It is convenient to write the t–J model without the local constraint in the following form:

$$H = -t \sum_{\langle i,j\rangle,\sigma} c^+_{i\sigma} c_{j\sigma} + J \sum_{\langle i,j\rangle} \left(S_i \cdot S_j - \tfrac{1}{4} n_i n_j\right) + U \sum_i n_{i\uparrow} n_{i\downarrow} \tag{1}$$

where $c^+_{i\sigma}$ creates an electron of spin σ on site i, $n_i = c^+_{i\sigma} c_{i\sigma}$ and $S_i = \frac{1}{2} c^+_{i\alpha} (\tau)_{\alpha\mu} c_{i\mu}$ are the electron density and spin operators, $\tau_{\alpha\mu} = \left(\tau^1_{\alpha\mu} \cdots \tau^3_{\alpha\mu}\right)$ are Pauli matrices and $\langle ij\rangle$ denotes nearest-neighbour sites.

The Hubbard term mimics the constraint on double occupancy in (1). By setting $U = \infty$ we recover the standard t–J model for $n \to 1$:

$$H = -t \sum_{\langle ij\rangle\sigma} \hat{c}^+_{i\sigma} \hat{c}_{j\sigma} + \tilde{J} \sum_{\langle ij\rangle} \left(S_i \cdot S_j - \tfrac{1}{4} n_i n_j\right)$$

with $\hat{c}_{i\sigma} = c_{i\sigma}(1 - n_{i-\sigma})$ and $\tilde{J} = J + 4t^2/U$ ($= J$ for $U \to \infty$). In a momentum space representation on the square lattice we obtain from (1)

$$\begin{aligned} H = \sum_{p\sigma} \varepsilon_p c^+_{p\sigma} c_{p\sigma} &+ \sum_{k_1 k_2 q \alpha \beta \gamma \mu} \frac{J(q)}{2} \left(\tau_{\alpha\beta}\tau_{\gamma\mu} - \delta_{\alpha\beta}\delta_{\gamma\mu}\right) c^+_{k_1\alpha} c^+_{k_2\gamma} c_{k_2-q\mu} c^+_{k_1+q\beta} \\ &+ U \sum_{k_1 k_2 q} c^+_{k_1\uparrow} c^+_{k_2\downarrow} c_{k_2-q\downarrow} c_{k_1+q\uparrow} \end{aligned} \tag{2}$$

where $\varepsilon_p = -2t(\cos p_x + \cos p_y)$, $J(q) = J(\cos q_x + \cos q_y)$. The lattice constant is set equal to one. The total interactions in the singlet (V_s) and triplet (V_t) channels for an electron pair $(p, -p)$ to be scattered to $(p', -p')$ are given by

$$\begin{cases} V_s(q) = U - 2J(q) \\ V_t(q) = 0 \end{cases} \tag{3}$$

where $q = p - p'$ is the transferred momentum. For the pairing problem it is convenient to expand $J(q)$ in (3) in a series with the eigenfunctions of the irreducible representations of the lattice symmetry group D_4. This yields

$$\cos q_x + \cos q_y \equiv \cos(p_x - p'_x) + \cos(p_y - p'_y) = \tfrac{1}{2}\big[(\cos p_x + \cos p_y)(\cos p'_x + \cos p'_y) + (\cos p_x - \cos p_y)(\cos p'_x - \cos p'_y)\big] + \sin p_x \sin p'_x + \sin p_y \sin p'_y. \quad (4)$$

The first term in square brackets corresponds to the A_1 identical representation (s-wave pairing), the second term to the B_1 representation ($d_{x^2-y^2}$ pairing) and the last two terms to the two-dimensional representation E (p-wave pairing). Note that for the singlet channel only the first two terms, which are in the square brackets, are relevant.

Substituting the formula (4) into (3) we obtain

$$\begin{cases} V(\text{s-wave channel}) = U - J(\cos p_x + \cos p_y)(\cos p'_x + \cos p'_y) \\ V(d_{x^2-y^2}\ \text{channel}) = -J(\cos p_x - \cos p_y)(\cos p'_x - \cos p'_y). \end{cases} \quad (5)$$

We are now ready to solve the **T**-matrix problem in both s-wave and d-wave channels. Let us consider first the s-wave channel. The Bethe–Salpeter integral equation for $T_{pp'}(E)$ has the following form:

$$T_{pp'}(E) = V_{pp'} + \iint V_{pp''} \frac{\mathrm{d}p''_x\,\mathrm{d}p''_y}{E + 4t(\cos p''_x + \cos p''_y)} T_{p''p'}(E) \quad (6)$$

where

$$V_{pp'} = U - J(\cos p_x + \cos p_y)(\cos p'_x + \cos p'_y) = U - J\varphi_p\varphi_{p'}$$

and $\varphi_p = \cos p_x + \cos p_y$. Here $(p, -p)$ and $(p', -p')$ are incoming and outgoing momenta in the particle–particle channel. We will use the following *ansatz* for $T_{pp'}$:

$$T_{pp'}(E) = T_1(E) + T_2(E)(\varphi_p + \varphi_{p'}) + T_3(E)\varphi_p\varphi_{p'}.$$

Then due to the orthogonality properties of functions φ_p it is possible to reduce the integral equations (6) to the system of three algebraic equations for T_1, T_2, T_3

$$\begin{cases} T_1 = U + UT_1I_0 + UT_2I_x \\ T_2 = -JT_1I_x - JT_2I_{xx} \\ T_3 = -J - JT_3I_{xx} - JT_2I_x \end{cases} \quad (7)$$

where

$$I_0 = \int_0^{2\pi} \frac{\mathrm{d}p_x\,\mathrm{d}p_y}{(2\pi)^2} \frac{1}{E + 4t(\cos p_x + \cos p_y)} = \frac{2}{\pi E} K\left(-\frac{8t}{E}\right).$$

K is the complete elliptic integral of the first order. The other integrals are

$$I_x = \int_0^{2\pi} \frac{\mathrm{d}p_x\,\mathrm{d}p_y}{(2\pi)^2} \frac{\cos p_x + \cos p_y}{E + 4t(\cos p_x + \cos p_y)} = \frac{1}{4t} - \frac{E}{4t} I_0$$

$$I_{xx} = \int_0^{2\pi} \frac{\mathrm{d}p_x\,\mathrm{d}p_y}{(2\pi)^2} \frac{(\cos p_x + \cos p_y)^2}{E + 4t(\cos p_x + \cos p_y)} = -\frac{E}{4t}\left(\frac{1}{4t} - \frac{E}{4t} I_0\right) = -\frac{E}{4t} I_x.$$

Solving (7) we arrive at results

$$T_1 = \frac{U(1+JI_{xx})}{(1-UI_0)(1+JI_{xx})+UJI_x^2}$$
$$T_2 = \frac{-UJI_x}{(1-UI_0)(1+JI_{xx})+UJI_x^2} \tag{8}$$
$$T_3 = \frac{-J(1-UI_0)}{(1-UI_0)(1+JI_{xx})+UJI_x^2}.$$

By analogy with the definition of the scattering length in quantum mechanics $\left(a = \lim_{p',p\to 0} f_{pp'},\ f \text{ is the scattering amplitude}\right)$, it is natural to define the **T**-matrix for two particles as a zero-momentum limit of $T_{pp'}(E)$. In the s-wave channel we obtain the result

$$T_s(E) = \lim_{p',p\to 0} T_{pp'}(E) = T_1(E) + 4T_2(E) + 4T_3(E)$$

which using (8) takes the form

$$T_s(E) = \frac{U(1+JI_{xx}) - 4UJI_x - 4J(1-UI_0)}{(1-UI_0)(1+JI_{xx})+UJI_x^2}. \tag{9}$$

For $U \to \infty$ the expression (9) can be simplified to yield

$$T_s(E) = \frac{1+JI_{xx} - 4JI_x + 4J\,I_0}{-I_0 - JI_0I_{xx} + JI_x^2}. \tag{10}$$

Taking into account the relations between I_0 on the one hand and I_x, I_{xx} on the other, we can rewrite (10) in the form

$$T_s(E) = \frac{1-(J/t)(E/16t+1) + (E^2/16t^2 + E/t + 4)JI_0}{-(1+JE/16t^2)I_0 + J/16t^2}.$$

Introducing the binding energy, $\widetilde{E} = E + 8t < 0$, we obtain for small values of $|\widetilde{E}|$

$$I_0 = -\frac{1}{8\pi t}\ln\frac{64t}{|\widetilde{E}|}.$$

As a result, neglecting terms of the order of $\widetilde{E}\ln\widetilde{E}$ which vanish as $\widetilde{E} \to 0$, we come to the following expression for $T_s(\widetilde{E})$:

$$T_s(|\widetilde{E}|) = \frac{(1-J/2t)}{[(1-J/2t)/8\pi t]\ln 64t/(|\widetilde{E}|) + J/16t^2}$$
$$= \frac{W(1-J/2t)}{(1/\pi)(1-J/2t)\ln 8W/|\widetilde{E}| + J/2t} \tag{11}$$

where $W = 8t$ is the bandwidth.

For $J \ll J_{c0}(=2t)$ the **T**-matrix $T_s(|\widetilde{E}|) = \pi W/\ln(8W/|\widetilde{E}|)$ is positive which corresponds to repulsion and coincides with the **T**-matrix obtained by Fukuyama *et al* [11] for the 2D Hubbard model at low electron density.

For $J = J_{c0}$, $T_s(|\widetilde{E}|) = 0$ and there is no interaction at all. However, for $J > J_{c0}$, $T_s(|\widetilde{E}|) < 0$ and this corresponds to attraction. Moreover, there exists now a bound state of two electrons [9, 10] on the empty lattice with the binding energy

$$|\widetilde{E}_b| = 8W \exp\left\{-\frac{\pi J}{(J - J_{c0})}\right\}.$$

Note that $|\widetilde{E}|$ tends to zero exponentially when J approaches J_{c0}. Finally for $J \gg J_{c0}$ we obtain $|\widetilde{E}_b| = 8W\exp\{-\pi\} \sim W$.

For the sake of completeness let us solve the **T**-matrix problem also in the d-wave channel. The corresponding Bethe–Salpeter equation has the following form:

$$T_{pp'}(E) = T_d\varphi_p\varphi_{p'} = -J\varphi_p\varphi_{p'} - JT_d\varphi_p\varphi_{p'}I_d \tag{12}$$

where $V_{pp'} = -J(\cos p_x - \cos p_y)(\cos p'_x - \cos p'_y)(= -J\varphi_p\varphi_{p'})$. The integral I_d is defined as

$$I_d = \int_0^{2\pi} \frac{dp''_x\, dp''_y}{(2\pi)^2} \frac{\varphi^2_{p''}}{E + 4t(\cos p''_x + \cos p''_y)}.$$

The solution of the equation (12) reads:

$$T_d = -\frac{J}{1 + JI_d}$$

and the calculation of I_d yields

$$I_d = \frac{-E}{16t^2} + \left(\frac{8}{\pi E} - \frac{E}{8\pi t^2}\right)K\left(-\frac{8t}{E}\right) + \frac{E}{4\pi t^2}\mathcal{E}\left(-\frac{8t}{E}\right)$$

where $K(-8t/E)$ and $\mathcal{E}(-8t/E)$ are complete elliptic integrals of the first and second kind (see [12]). Introducing again the binding energy $\widetilde{E}(= E + 8t)$ we obtain for small values of $|\widetilde{E}|$ (or, $E \to -8t$)

$$K\left(-\frac{8t}{E}\right) = \frac{1}{2}\ln\frac{64t}{|\widetilde{E}|}$$

$$\mathcal{E}\left(-\frac{8t}{E}\right) = 1 + \frac{|\widetilde{E}|}{16t}\left(\ln\frac{64t}{|\widetilde{E}|} - 1\right).$$

Then neglecting terms of the order of $\widetilde{E}^2$ we obtain the limiting value

$$I_d = -\frac{1}{2t}\left(\frac{4}{\pi} - 1\right) + \frac{|\widetilde{E}|}{16t^2}\left(1 - \frac{2}{\pi}\right).$$

As a result

$$T_d(|\widetilde{E}|) = \frac{-J}{1 - (J/2t)(4/\pi - 1) + (J|\widetilde{E}|/16t^2)(1 - 2/\pi)}. \tag{13}$$

It follows that only for $J > J_{cd} = 2t/(4/\pi - 1) \simeq 6t$ does a bound state with d-wave symmetry exist for two particles on the empty lattice. The binding energy of this state is given by

$$|\widetilde{E}_b| = \frac{(J - J_{cd})}{J} \frac{W2t}{J_{cd}(1 - 2/\pi)} \simeq W \frac{J - J_{cd}}{J}.$$

Note that the threshold J_{cd} for a d-wave bound state is much larger than the threshold J_{c0} for an s-wave bound state. Also $|\widetilde{E}_b|$ tends to zero linearly and not exponentially when J approaches J_{cd}. For $J \gg J_{cd}$, the binding energy saturates and $|\widetilde{E}_b| \to W$.

3. The Cooper pairing problem

We proceed now from the two-particle problem to the problem of Cooper pairing of two particles in the presence of a filled Fermi sea. We know that in two dimensions it is obligatory for an s-wave Cooper pair to be accompanied by a bound s-wave state of two particles in the empty lattice [13–15]. It follows that $J = J_{c0} (\equiv 2t)$ serves as a threshold for s-wave Cooper pairing. At the same time, to form p-wave and d-wave Cooper pairs, the existence of the corresponding bound state is not obligatory.

It follows that p-wave Cooper pairs can exist for arbitrary small values of J. Also d-wave Cooper pairs can be realized below the threshold for the bound state, i.e. for $J < J_{cd} (\simeq 6t)$.

A knowledge of the **T**-matrices for s-wave and d-wave bound states ($\widetilde{E} < 0$) allows us to obtain in straightforward manner the expressions for mean field superconducting critical temperatures. The only modifications we must introduce in the expressions (11, 13) for $T_s(|\widetilde{E}|)$ and $T_d(|\widetilde{E}|)$ are the following. We must replace $\widetilde{E} < 0$ by $\widetilde{E} = 2\varepsilon_F (= p_F^2/m > 0)$ (ε_F is the Fermi energy) and we must add imaginary parts to the denominators of (11) and (13) which are absent in the bound-state problem. To be more specific, for $\widetilde{E} < 0$

$$\mathrm{Im}\, I_0 = \mathrm{Im}\, I_d = 0$$

but for $\widetilde{E} = 2\varepsilon_F > 0$

$$\mathrm{Im}\, I_0 = \int_0^{2\pi} \frac{\mathrm{d}p_x\, \mathrm{d}p_y}{4\pi} \delta\big[\widetilde{E} + 4t(\cos p_x + \cos p_y) - 8t\big] = \frac{1}{8\pi t}\pi = \frac{m}{4} \neq 0 \qquad m = 1/2t$$

$$\mathrm{Im}\, I_d = \int_0^{2\pi} \frac{\mathrm{d}p_x\, \mathrm{d}p_y}{4\pi} (\cos p_x - \cos p_y)^2 \delta\big[\widetilde{E} + 4t(\cos p_x + \cos p_y) - 8t\big] = \frac{\widetilde{E}^2}{W^2}\frac{1}{J_{cd}}$$
$$= \left(\frac{2\varepsilon_F}{W}\right)^2 \frac{1}{J_{cd}}.$$

As a result, we obtain the following forms for the **T**-matrices in the positive energy sector:

$$T_s(\widetilde{E} = 2\varepsilon_F) = \frac{8\pi t}{\ln(4W/E_F) + \pi J (J_{c0} - J)^{-1} + \mathrm{i}\pi}$$
$$T_d(\widetilde{E} = 2\varepsilon_F) = \frac{-J J_{cd}}{J_{cd} - J + 2\varepsilon_F J/W + \mathrm{i}J(2\varepsilon_F/W)^2}. \qquad (14)$$

The gas parameter for the s-wave channel is given by

$$f_0 = \frac{mT_s(\widetilde{E} = 2\varepsilon_F)}{4\pi} = \frac{1}{\ln(4W/\varepsilon_F) - \pi J(J - J_{c0})^{-1} + i\pi} . \quad (15)$$

Note that for $\widetilde{E} > 0$ there is no pole in the expression for $T_s(\widetilde{E})$ and $T_d(\widetilde{E})$. For $\varepsilon_F = \frac{1}{2}|\widetilde{E}_b|$ we are in a resonant situation. It follows that

$$T_s(2\varepsilon_F = |E_b|) = -8it \qquad T_d(2\varepsilon_F = |E_b|) = iJ_{cd}W^2/4\varepsilon_F^2$$

and the **T**-matrices reach the unitary limit.

Let us proceed now to the calculation of critical temperatures. For $J \ll J_{c0}$, the gas parameter $f_0 = (\ln(4W/\varepsilon_F))^{-1} > 0$ and s-wave Cooper pairing is absent. However, even for these values of J, p-wave pairing and d_{xy} pairing are possible [6, 7]. To obtain d_{xy} pairing it is necessary to calculate the irreducible bare vertex for Cooper channel $\widetilde{\Gamma}_{pp'}$ up to the second order in f_0. For low density ($p_F \ll 1$) this yields an attraction the in B_2 (d_{xy}) channel (see [6] for more details).

$$\widetilde{\Gamma}_{B_2} \equiv \widetilde{\Gamma}_{d_{xy}} = -f_0^2 \frac{p_F^4}{320} (\sin p_x \sin p_y)(\sin p_x' \sin p_y').$$

The mean field critical temperature for d_{xy} pairing reads

$$T_c^{xy} \sim \varepsilon_F \exp\left\{-\frac{320}{f_0^2\, p_F^4}\right\}. \quad (16)$$

To obtain p-wave pairing it is necessary to take into account third-order diagrams for $\Gamma_{pp'}$. This yields an attraction in the E (p-wave) channel (see [7] for more details)

$$\widetilde{\Gamma}_E \equiv \widetilde{\Gamma}_{p\text{ wave}} = -4.1 f_0^3 \sin p_x \sin p_x'.$$

The mean field critical temperature for p-wave pairing is given by

$$T_c^p \sim \varepsilon_F \exp\left\{-\frac{1}{4.1 f_0^3}\right\}. \quad (17)$$

For low density $\widetilde{\Gamma}_E > \widetilde{\Gamma}_{B_2}$ because $f_0^3/f_0^2\, p_F^4 = f_0/p_F^4 \sim 1/p_F^4 \ln(1/p_F) \gg 1$. That is why for $J \ll J_{c0}$ and $n \to 0$ $T_c^p > T_c^{xy}$ and the p-wave pairing will be stabilized.

To have the full picture for $J < J_{c0}$ we must also calculate the critical temperature for $d_{x^2-y^2}$ pairing. The corresponding Bethe–Salpeter equation for the total vertex Γ_d in the Cooper channel has the following form for this d-wave pairing:

$$\Gamma_d = -J - \frac{J\Gamma_d}{(2\pi)^2} \int_0^{2\pi} \frac{(\cos p_x - \cos p_y)^2 dp_x\, dp_y}{-4t(\cos p_x + \cos p_y) - 2\mu} \operatorname{th}\left(\frac{-2t(\cos p_x + \cos p_y) - \mu}{2T}\right) \quad (18)$$

where T is the temperature and $\mu = -4t + \varepsilon_F$.

After the standard renormalization procedure which corresponds to the addition and subtraction from (16) of the integral

$$\frac{J\Gamma_d}{(2\pi)^2} \int_0^{2\pi} \frac{(\cos p_x - \cos p_y)^2}{-4t(\cos p_x + \cos p_y) + 8t} dp_x dp_y$$

we obtain for small density

$$\Gamma_d = \frac{-J}{1 - J/J_{cd} - J p_F^4 (32\pi t)^{-1} \ln(\varepsilon_F / T_c^{x^2-y^2})}$$

and accordingly

$$T_c^{x^2-y^2} \sim \varepsilon_F \exp\left\{ -\frac{(J_{cd} - J) 32\pi t}{J p_F^4 J_{cd}} \right\} = \varepsilon_F \exp\left\{ -\frac{32\pi t}{T_d p_F^4} \right\}. \tag{19}$$

Introducing electron density $n = p_F^2/2\pi$ (n is measured in units of a half-filled band, i.e. n is the electron density/site) we can rewrite the formulae (17) and (19) for the mean field transition temperatures for $d_{x^2-y^2}$- and p-wave pairing in the following forms:

$$\begin{aligned} \ln\left(\frac{\varepsilon_F}{T_c^{x^2-y^2}}\right) &= \frac{8t}{\pi T_d n^2} \qquad T_d = \frac{J J_{cd}}{J_{cd} - J} \\ \ln\left(\frac{\varepsilon_F}{T_c^p}\right) &= \frac{1}{4.1}\left(\ln\left(\frac{16}{\pi n}\right) + \frac{J\pi}{J_{c0} - J}\right)^3 = \frac{1}{4.1 f_0^3}. \end{aligned} \tag{20}$$

A comparison of the critical temperatures yields

$$\text{for } n = 0.2 \qquad T_c^p > T_c^{x^2-y^2} \qquad \text{for } J < t$$

$$\text{for } n = 0.5 \qquad T_c^p > T_c^{x^2-y^2} \qquad \text{for } J < 0.6t.$$

It follows that the region of the J–n phase diagram ($n \leqslant 0.5$; $J < J_{c0}$), which in numerical calculations of Dagotto and co-workers [8] is called a paramagnetic phase, in reality has p-wave superconducting pairing for smaller values of J and $d_{x^2-y^2}$ pairing for larger J.

Let us proceed now to the case of $J > J_{c0}$. In this case an s-wave pairing takes place below the critical temperature (see [13–15]).

$$T_c^s \sim \varepsilon_F \exp\left\{ -\frac{1}{| f_0 |} \right\} \tag{21}$$

$f_0 = \{\ln(4W/\varepsilon_F) - J\pi/(J - J_{c0}) + i\pi\}^{-1}$; $|f_0|$ denotes the absolute magnitude of f_0.

Everywhere, except for a narrow region close to the resonance ($\varepsilon_F \to |\widetilde{E}_b|/2 = 4W \exp\{-J\pi/(J - J_{c0})\}$), we can rewrite (21) in the form

$$T_c^s \sim \sqrt{2\varepsilon_F |\widetilde{E}_b|} = \frac{8\sqrt{2}\,\varepsilon_F}{p_F} \exp\left\{ -\frac{J\pi}{2(J - J_{c0})} \right\}. \tag{22}$$

Formula (22) describes both the case of weakly bound Cooper pairs for $\varepsilon_F \gg |E_b|$ and tightly bound dimers or 'bipolarons' [16, 17] for $\varepsilon_F \ll |\widetilde{E}_b|$. The chemical potential of the system, found by Miyake [13] from the self-consistent approach of Leggett [18], reads $\mu = \varepsilon_F - |E_b|/2$ and describes for $\mu > 0$ $(\varepsilon_F \gg |\widetilde{E}_b|)$ the superconducting Fermi gas and for $\mu < 0$ $(\varepsilon_F \ll |\widetilde{E}_b|)$ the Bose liquid of bipolarons. The energy of the system also smoothly interpolates between these two limits

$$\frac{E_s^{BCS} - E_N}{N} \sim -\frac{\Delta^2}{4\varepsilon_F} \sim -\frac{T_c^2}{4\varepsilon_F} \sim -\frac{\left(\sqrt{2\varepsilon_F |\widetilde{E}_b|}\right)^2}{4\varepsilon_F} \sim -\frac{| \widetilde{E}_b |}{2} \sim \frac{E_s^{bipol} - E_N}{N} \tag{23}$$

where Δ is a superconductive gap, N is a total number of particles and $E_N = (\varepsilon_F/2)N$ is an energy of a normal state.

Formula (23) means that for $\varepsilon_F \ll |\widetilde{E}_b|$ the energy of a superconducting state E_s becomes negative and we pass from the gas phase to the liquid phase. Increasing J or $|\widetilde{E}_b|$ for a fixed value of ε_F (i.e., for fixed density) we finally will be in a situation where the dimers will condense in droplets, forming a non-homogeneous state. The numerical calculations of Dagotto and co-workers give the value $J_{PS} = 3.8t$ $(n \to 0)$ for the phase-separation instability. Note that analytical calculations of the phase-separation threshold require at least the evaluation of a four-particle vertex (describing the interaction between two dimers) and is rather cumbersome.

Let us now compare the temperatures of s-wave and d-wave pairing in the region $n < 0.5$ and $J_{c0} < J < J_{PS}$. Comparison of formulae (19) and (22) for $T_c^{x^2-y^2}$ and T_c^s yields

$$\text{for } J = 2.5t \qquad T_c^{x^2-y^2} > T_c^s \qquad \text{for } n > 0.29$$

$$\text{for } J = 3t \qquad T_c^{x^2-y^2} > T_c^s \qquad \text{for } n > 0.30$$

and for J close to phase separation $T_c^{x^2-y^2} > T_c^s$ also for $n > 0.3$.

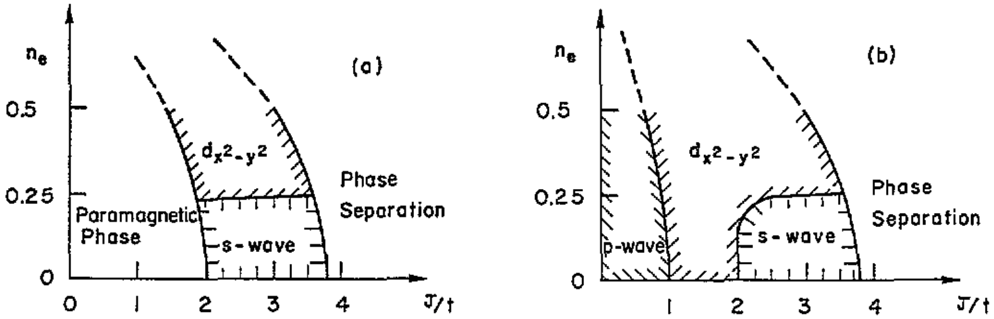

Figure 1. The phase diagram of the two-dimensional t–J model showing the regions of stability of s-wave, p-wave and $d_{x^2-y^2}$ pairing as functions of the electron density (n_e) and the ratio (J/t) obtained in (a) the numerical calculations by Dagotto and co-workers [8] and (b) this work using an analytic low-density expansion.

It follows that, in agreement with numerical calculations of Dagotto and co-workers [8], when $J > J_{c0}$ an s-wave pairing is stabilized for smaller densities while for larger densities d-wave pairing is more stable. Summing up the comparison between s-wave and d-wave temperatures on one hand and between d-wave and p-wave temperatures on the other hand we conclude that for $n > 0.3$ and for $J > t$, d-wave pairing is stabilized. Moreover when we increase the density it is stabilized even for smaller values of J. For instance d-wave pairing is stable for $n = 0.5$, $J = 0.6t$. It follows that the increase of electron density favours a d-wave pairing. In figure 1, we show the phase diagram obtained by numerical and analytic methods. If we make a rough extrapolation to the region of parameters relevant for high-T_c materials, i.e. $J \sim \frac{1}{2}t$, $n \sim 0.85$, we obtain from (19)

$$T_c^{x^2-y^2}\left(n = 0.85;\ J/t = \tfrac{1}{2}\right) \sim \varepsilon_F \exp\{-7\} \sim \varepsilon_F \times 10^{-3} \sim 10\ \text{K for } \varepsilon_F \sim 10^4\ \text{K}.$$

4. Conclusions

In conclusion, we would like to repeat that as expected, the two-dimensional t–J model at low electron density is equivalent for $J \ll t$ to the two-dimensional Hubbard model with respect to superconductive pairing. It follows that for $J \ll t$ the t–J model is unstable towards p-wave superconductive pairing. For $J > t$ the leading instability at low density in the t–J model is towards s-wave or $d_{x^2-y^2}$ pairing. Moreover, we find good agreement between analytic and numerical calculations for the location of s- and d-wave pairing regions, and the phase boundary between them in the J–n phase diagram.

Acknowledgment

The authors are grateful to H Monien for helpful discussions.

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